Chapter 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

De Moivre's theorem:

1) $\left(\cos\theta+\sin\theta\right)^{n}=\cos n\theta+i\sin\theta$, **where 'n' is an integer.**

2) $\left(\cos\theta+i\sin\theta\right)^{-n}=\cos\left(n\theta\right)-i\sin\left(n\theta\right)$

3) $\left(\cos\theta-i\sin\theta\right)^{n}=\cos n\theta-i\sin\left(n\theta\right)$

4) $\left(\cos\theta-i\sin\theta\right)^{-n}=\cos n\theta+i\sin n\theta$

Note:

1) $\cos\theta+i\sin\theta=\operatorname{cis}\theta$

2) $\left(\cos\theta+i\sin\theta\right)^{n}=\operatorname{cis}\left(n\theta\right)$

3) $\cos\theta+i\sin\theta=\dfrac{1}{\cos\theta-i\sin\theta}$

4) $\operatorname{cis}\theta_1.\operatorname{cis}\theta_2=\operatorname{cis}\left(\theta_1+\theta_2\right)$

5) $\dfrac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2}=\operatorname{cis}\left(\theta_1-\theta_2\right)$

Problem:

If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma$, **prove that**

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}=\sin^2\alpha+\sin^2\beta+\sin^2\gamma.$$

Sol.

Let a = cis α

b = cis β

c = cis γ

a + b + c = cis(α) + cis(β) + cis (γ)

$$= (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$$

$$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$

a + b + c = 0 + i. 0 = 0

Also, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{\text{cis}\alpha} + \frac{1}{\text{cis}\beta} + \frac{1}{\text{cis}\gamma}$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \text{cis}(-\alpha) + \text{cis}(-\beta) + \text{cis}(-\gamma)$$

$$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$$

= 0 – i.0 = 0

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

a + b + c = 0

Squaring on both sides,

$(a + b + c)^2 = 0$

$a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

$a^2 + b^2 + c^2 = -2abc$ $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

$a^2 + b^2 + c^2 = -2abc(0)$ $\left(\because \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0\right)$

$$(\text{cis}\alpha)^2 + (\text{cis}\beta)^2 + (\text{cis}\lambda)^2 = 0$$

$$(\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta) + (\cos 2\gamma + i\sin 2\gamma) = 0 = 0 + i.0$$

Equating real part on R.H.S

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$

$$2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$$

$$2\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right) = 3$$

$$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$$

Again, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$$1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$$

$$3 - 2\left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right) = 0$$

$$3 = 2\left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right)$$

$$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 0$$

Hence proved.