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Chapter 07

INDEFINITE INTEGRAL

TYPES OF INTEGRATION:

- If $b^2-4ac<0$, then
  $$\int\frac{dx}{ax^2+bx+c}=\frac{2}{\sqrt{4ac-b^2}}Tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b}}\right)+k$$
- If $b^2-4ac<0$, then
  $$\int\frac{Ax+B}{ax^2+bx+c}dx=\frac{A}{2a}\log\left|ax^2+bx+c\right|+\frac{2(2aB-bA)}{A\sqrt{4ac-b^2}}Tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+k$$
- If $b^2-4ac>0$, then $\frac{Ax+B}{ax^2+bx+c}$ has partial fractions.

  After resolving partial fractions we can find
  $$\int\frac{Ax+B}{ax^2+bx+c}dx.$$
- If $a>0, b^2-4ac<0,$ then
  $$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}Sinh^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$$
- If $a>0, b^2-4ac>0,$ then
  $$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}Cosh^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)$$
- If $a<0, b^2-4ac>0,$ then
  $$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{-a}}Sin^{-1}\left(\frac{-2ax-b}{\sqrt{b^2-4ac}}\right)$$
- $$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx=\frac{A}{a}\sqrt{ax^2+bx+c}+\frac{2aB-bA}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$
- $$\int\sqrt{ax^2+bx+c}\,dx=\frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a}+\frac{4ac-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$$
- $$\int\frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx=\frac{ac+bd}{c^2+d^2}x+\frac{ad-bc}{c^2+d^2}\log\left|c\cos x+d\sin x\right|+c$$

- If $a > b$ then

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} Tan^{-1}\left[\frac{\sqrt{a-b}}{a+b}\tan\frac{x}{2}\right]+c$$

- If $a < b$ then

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}} \log\left|\frac{\sqrt{b+a}+\sqrt{b-a}\tan(x/2)}{\sqrt{b+a}-\sqrt{b-a}\tan(x/2)}\right|+c$$

- If $a>0, a>b$, then

$$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} Tan^{-1}\left[\frac{b+a\tan(x/2)}{\sqrt{a^2-b^2}}\right]+c$$

- If $0<a<b$, then

$$\int \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}} \log\left|\frac{a\tan(x/2)+b-\sqrt{b^2-a^2}}{a\tan(x/2)+b+\sqrt{b^2-a^2}}\right|+c$$

- If $a>b, a^2>b^2+c^2$ then

$$\int \frac{dx}{a+b\cos x+c\sin x} = \frac{2}{\sqrt{a^2-b^2-c^2}} Tan^{-1}\left[\frac{(a-b)\tan(x/2)+c}{\sqrt{a^2-b^2-c^2}}\right]+k$$

- If $a>b, a^2<b^2+c^2$, then

$$\int \frac{dx}{a+b\cos x+c\sin x} = \frac{1}{\sqrt{b^2+c^2-a^2}} \log\left|\frac{(a-b)\tan(x/2)+c-\sqrt{b^2+c^2-a^2}}{(a-b)\tan(x/2)+c+\sqrt{b^2+c^2-a^2}}\right|+c$$

- If $a<b$, then

$$\int \frac{dx}{a+b\cos x+c\sin x} = \frac{-1}{\sqrt{b^2+c^2-a^2}} \log\left|\frac{(b-a)\tan(x/2)-c-\sqrt{b^2+c^2-a^2}}{(b-a)\tan(x/2)-c+\sqrt{b^2+c^2-a^2}}\right|+k$$

Type- I:

a) $\int \frac{px+q}{ax^2+bx+c}\,dx$ (or) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

(or) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

then put $px+q = A\frac{d}{dx}\left(ax^2+bx+c\right)+B$

Example-3:

Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx\,,\,x\in R$

Solurion:

Let $2x+5=A\frac{d}{dx}\left(x^2-2x+10\right)+B$

$2x+5=A\left(2x-2\right)+B$ ....(1)

Comparing on both sides $\cancel{2}=\cancel{2}A\Rightarrow A=1$

$5=-2A+B\ =-2(1)+B$

$B=7$

$$\int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx=\int \frac{1(2x-2)+7}{\sqrt{x^2-2x+10}}\,dx$$

$$=\int \frac{2x-2}{\sqrt{x^2-2x+10}}\,dx+\int \frac{7}{\sqrt{x^2-2x+10}}\,dx$$

$$=\int \frac{d\left(x^2-2x+10\right)}{\sqrt{x^2-2x+10}}+\int \frac{7}{\sqrt{x^2-2x+(1)+9}}\,dx$$

$$=2\sqrt{x^2-2x+10}+7\int \frac{1}{\sqrt{(x-1)^2+(3)^2}}\,dx$$

$$\left[\because\ \int \frac{1}{\sqrt{x^2+a^2}}\,dx=\sinh^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$=2\sqrt{x^2-2x+10}+7\sinh^{-1}\left(\frac{x-1}{3}\right)+C.$$

Type- II:

$$\int \frac{px+q}{\sqrt{ax+b}}\,dx$$

(or) $$\int \frac{\sqrt{ax+b}}{px+q}\,dx$$

(or) $$\int (px+q)\sqrt{ax+b}\,dx$$

(or) $$\int \frac{dx}{(px+q)\sqrt{ax+b}}$$

$$\boxed{\text{Put } ax+b=t^2 \Rightarrow dx=\frac{1}{a}2t\,dt}$$

Example-4:

$$\int \frac{1}{(x+2)\sqrt{x+1}}\,dx\,;\ x\in(-1,\infty)$$

Solution:

Put $x+1=t^2 \Rightarrow dx=2t\,dt$

$$\int \frac{1}{(x+1+1)\sqrt{x+1}}\,dx=\int \frac{1\times 2t}{\left(t^2+1\right)\sqrt{t^2}}dt$$

$$=2\int \frac{t\,dt}{\left(t^2+1\right)t}=2\tan^{-1}(t)+C=2\,\mathrm{Tan}^{-1}\left(\sqrt{x+1}\right)+C\,.$$

Type-III:

$$\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}\,dx\,;$$

put $px+q=\dfrac{1}{t}$.

Example-5:

Evaluate $\displaystyle\int \frac{1}{(1+x)\sqrt{3+2x-x^2}}\,dx\ ,\ x\in(-1,3)$

Solution:

Put $1+x=\dfrac{1}{t}\Rightarrow x=\dfrac{1}{t}-1\Rightarrow dx=-\dfrac{1}{t^2}dt$

Consider,

$$3+2x-x^2=3+2\left(\frac{1}{t}-1\right)-\left(\frac{1}{t}-1\right)^2$$

$$=3+\frac{2}{t}-2-\left\{\frac{1}{t^2}+1-\frac{2}{t}\right\}$$

$$=\cancel{3}+\frac{2}{t}-\cancel{2}-\frac{1}{t^2}-\cancel{1}+\frac{2}{t}$$

$$3+2x-x^2=\frac{2+2}{t}-\frac{1}{t^2}=\frac{4}{t}-\frac{1}{t^2}=\frac{4t-1}{t^2}$$

Now, $$\int \frac{1}{(1+x)\sqrt{3+2x-x^2}}\,dx=\int \frac{-1/\cancel{t^2}\,dt}{\dfrac{1}{\cancel{t}}\sqrt{\dfrac{4t-1}{t^2}}}$$

$$=\int \frac{-1}{t\dfrac{\sqrt{4t-1}}{t}}\,dt$$

$$= -\int (4t-1)^{-1/2} dt$$

$$= -\frac{1}{4}\frac{(4t-1)^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= -\frac{1}{\cancel{4}^{2}}\frac{(4t-1)^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + C$$

$$= -\frac{1}{2}\left(4\left(\frac{1}{x+1}\right)-1\right)^{1/2} + C$$

$$= -\frac{1}{2}\left(\frac{4-x-1}{x+1}\right)^{1/2} + C$$

$$= -\frac{1}{2}\sqrt{\frac{3-x}{1+x}} + C.$$

Type-IV: (Trignometry Rational Form)

Example-6:

Evaluate $\int \frac{1}{5+4\cos 2x} dx$

Solution:

Put $\tan x = t$ then $dx = \frac{dt}{1+t^2}$, $\cos 2x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

Now,

$$\int \frac{1}{5+4\cos 2x} dx = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{dt}{1+t^2}$$

$$= \int \frac{\cancel{1+t^2}}{5(1+t^2)+4(1-t^2)} \cdot \frac{dt}{\cancel{1+t^2}}$$

$$= \int \frac{dt}{5+5t^2+4-4t^2}$$

$$=\int \frac{dt}{9+t^2}$$

$$=\int \frac{1}{(3)^2+(t)^2}\,dt=\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$$

$$=\frac{1}{3}\tan^{-1}\left(\frac{1}{3}.\tan x\right)+C$$

Example-7:

$$\int \frac{1}{3\cos x+4\sin x+6}\,dx$$

Solution:

$\tan(x/2)=t$ then $dx=\dfrac{2dt}{1+t^2}$, $\sin x=\dfrac{2t}{1+t^2}$, $\cos x=\dfrac{1-t^2}{1+t^2}$

$$=\int \frac{1}{3\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)+6}.\frac{2dt}{1+t^2}$$

$$=\int \frac{\cancel{1+t^2}}{3-3t^2+8t+6(1+t^2)}.\frac{2\,dt}{\cancel{1+t^2}}$$

$$=\int \frac{2\,dt}{3-3t^2+8t+6+6t^2}$$

$$=\int \frac{2\,dt}{3t^2+8t+9}$$

$$=\frac{2}{3}\int \frac{dt}{\left(t^2+\frac{8}{3}t+3\right)}$$

$$=\frac{2}{3}\int \frac{dt}{\left(\underbrace{(t)^2+2(t).\left(\frac{4}{3}\right)+\frac{16}{9}}-\frac{16}{9}+3\right)}$$

$$=\frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2-\frac{16+27}{9}}$$

$$=\frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{2}{\cancel{3}} \cdot \frac{1}{\frac{\sqrt{11}}{\cancel{3}}} . \tan^{-1}\left(\frac{t+\frac{4}{3}}{\frac{\sqrt{11}}{3}}\right) + C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{3t+4}{\sqrt{11}}\right) + C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{3\left(\tan\frac{x}{2}\right)+4}{\sqrt{11}}\right) + C$$

Type-V:

Example-8:

Evaluate $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}\, dx$

Solution:

Let $2\cos x + 3\sin x = A\frac{d}{dx}(4\cos x + 5\sin x) + B(4\cos x + 5\sin x)$

$$2\cos x + 3\sin x = A[-4\sin x + 5\cos x] + B(4\cos x + 5\sin x) \text{..........(1)}$$

Comparing $\cos x$ and $\sin x$ coefficients.

$$2 = 5A + 4B \rightarrow (2)$$

$$3 = -4A + 5B \rightarrow (3)$$

Multiply equation(2) with number ‘-4’.

Multiply equation(3) with number ‘5’.

Solving both equations.

$$\begin{array}{l} -8 = -\ \cancel{20A} - 16B \\ {}_{-}15 = +_{-}\cancel{20A} +_{-}\ 25B \\ \hline +23 = \quad +41B \end{array}$$

$$B = \frac{23}{41};\ A = -\frac{2}{41}$$

$$\therefore \int \frac{2\cos x+3\sin x}{4\cos x+5\sin x}\,dx=\int \frac{\left(\frac{-2}{41}\right)(-4\sin x+5\cos x)+\left(\frac{23}{41}\right)(4\cos x+5\sin x)}{4\cos x+5\sin x}\,dx$$

$$=\frac{-2}{41}\int \frac{-4\sin x+5\cos x}{4\cos x+5\sin x}\,dx+\frac{23}{41}\int \cancel{\frac{4\cos x+5\sin x}{4\cos x+5\sin x}}\,dx$$

$$=\frac{-2}{41}\log\left|4\cos x+5\sin x\right|+\frac{23}{41}x+C$$

REDUCTION FORMULAE:

- If $I_n=\int x_n\ e^{ax}\ dx$, then $\mathrm{I_n}=\frac{e^{ax}}{a}x^n-\frac{n}{a}I_{n-1}$

  where n is a positive integer

- If $I_n=\int \sin^n dx$, then $\mathrm{I_n}=\frac{-\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}I_{n-2}$,

  where n is a positive integer

- If $I_n=\int \cos^n dx$, then $\mathrm{I_n}=\frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}I_{n-2}$.

- If $I_n=\int \tan^n dx$, then $\mathrm{I_n}=\frac{\tan^{n-1}x}{n-1}-I_{n-2}$

- If $I_n=\int \cot^n dx$, then $\mathrm{I_n}=\frac{-\cot^{n-1}x}{n-1}-I_{n-2}$

- If $I_n=\int \sec^n dx$, then $\mathrm{I_n}=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}I_{n-2}$

- If $I_n=\int (\log x)^n\,dx$, then $\mathrm{I_n}=x(\log x)^n-nI_{n-1}$

- If $\mathrm{I}_{m,n}=\int \sin^m x\cos^n x dx$ then

$$I_{m,n}=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}I_{m,n-2}=-\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}+\frac{m-1}{m+n}I_{m-2,n}$$

Example-1:

If $I_n=\int\sin^n x\,dx$, then prove that

$$I_n=\frac{-\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}I_{n-2}$$

Solution:

$$I_n=\int\sin^n x\,dx\text{.......(1)}$$

$$I_n=\int\underbrace{\sin^{n-1}x}_{\mathrm{I}}\,\underbrace{\sin x}_{\mathrm{II}}\,dx$$

$$=\sin^{n-1}x\int\sin x\,dx-\int\left[(n-1)\sin^{n-1-1}x\cos x\int\sin x\,dx\right]$$

$$=-\sin^{n-1}x\cos x-(n-1)\int\left[\sin^{n-2}x\cos x(-\cos x)\right]dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cos^2 x\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\left(1-\sin^2 x\right)dx$$

$$I_n=-\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\,dx-(n-1)\int\sin^n x\,dx$$

$$I_n=-\sin^{n-1}x\cos x+(n-1)I_{n-2}-(n-1)I_n$$

$$I_n + (n-1)I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

$$I_n[\cancel{1} + n - \cancel{1}] = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$. Hence proved.

Example-2:

If $I_n = \int \tan^n x \, dx$ then prove that $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$. And deduce $\int \tan^6 x \, dx$.

Solution:

Let, $I_n = \int \tan^n x \, dx$

$$= \int \tan^{n-2} x . \tan^2 x \, dx$$

$$= \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \int (\tan x)^{n-2} . d(\tan x) - \int \tan^{n-2} x \, dx$$

$$\left[\int [f(x)]^n . f^1(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + 1 \right]$$

$$I_n = \frac{(\tan x)^{n-2+1}}{n-2+1} - I_{n-2}$$

$$I_n = \frac{\left(\tan^{n-1} x\right)}{n-1} - I_{n-2}$$

$$\int \tan^6 x\,dx = I_6 = \frac{\tan^5 x}{5} - I_4$$

$$= \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - I_2\right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - I_0 + C$$

$$\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C\,.$$